Definitions and key facts for section 1.1

A **linear equation** in the variables $x_1, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots a_nx_n = b$$

where b and the **coefficients** $a_1, \ldots, a_n$ are real of complex numbers.

A **system of linear equations** (or **linear system**) is a collection of one or more linear equations in the same variables. For example, here is a linear system of m equations in the n variables $x_1, \ldots, x_n$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n &= b_m \end{aligned}$$

A **solution** of a system is a list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, \ldots, s_n$ are substituted for $x_1, \ldots, x_n$, respectively.

The **solution set** of a system is the set of all possible solutions to that system.

Two linear systems are **equivalent** if they have the same solution set.

The **existence and uniqueness questions** that naturally arise when regarding a linear system are

- **Existence**: does a solution to the linear system *exist?*
- **Uniqueness**: is the solution of the linear system *unique?*

We say a linear system is **consistent** if it has a solution and **inconsistent** otherwise.

Fact: the three types of solution behavior for a linear system
A linear system has

1. no solution, or
2. exactly one solution, or
3. infinitely many solutions.

A **matrix** is a rectangular array of numbers. For example $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{bmatrix}$ is a matrix.

The **coefficient matrix** of a system is the matrix whose ith column consist of the coefficients of the ith variable x_i in the system. For example, the previous matrix A is the coefficient matrix of

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 \qquad - 5x_3 &= 10 \end{aligned}$$

The **augmented matrix** of a system consists of the coefficient matrix along with an additional column containing the constants on the right sides of the equations in the system. For example, the following matrix is the augmented matrix of the previous system

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$$

The three **elementary operations** which yield an equivalent linear system are

1. **Replacement:** Replace one equation by the sum of itself and a multiple of another equation.

2. **Interchange:** Interchange the positions of two equations.

3. **Scaling:** Multiply both sides of an equation by a nonzero constant.

The three **elementary row operations** which yield a row equivalent matrix are

1. **Replacement:** Replace one row by the sum of itself and a multiple of another equation. In symbols: $R1 : R1 + 5R2$.

2. **Interchange:** Interchange two rows. In symbols: $R1 \leftrightarrow R4$.

3. **Scaling:** Multiply all entries of a row by a nonzero constant. In symbols: $-6R1$.

By **row equivalent**, we mean any two matrices that can be transformed into one another by a sequence of elementary row operations.